THE HODOGRAPH METHOD IN ELECTRODYNAMICS OF CONTINUOUS NONLINEARLY CONDUCTING MEDIA

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Iu.P. EMETS
(Kiev)

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In a wide class of conductors, the presence of strong electric fields disrupts the linearity of the relationship between the current density and the electric field strength. Plasma and semiconductors exhibit the most observable deviations from the Ohm's law. This lack of linearity results from the fact that the current carriers assume nonequilibrium states. During the collisions, the electrons transfer their energy to the ions, molecules e.t.c. relatively slow because of their small mass. At the same time, they possess arbitrarily large free paths and can, therefore absorb large amount of energy from the electric field between successive collisions. Thus their temperature may differ appreciably from the equilibrium temperature due to the heating effect of the field.

Since initial equations are nonlinear, we may experience a difficulty in determining the current distribution in media with these properties. However, using the hodograph transformation we can linearise the electrodynamic equations. This, and use of approximate methods, give us an effective procedure for computing the electric fields in nonlinearly conducting media.

1. Equations of the force and potential functions of an electric current. Equation of state will be our starting point in constructing electrodynamic relations for the dense, nonlinearly conducting media. Its general form is

$$F(\mathbf{j}, \mathrm{E}, \sigma, x, y, z, t, \ldots) = 0 \qquad (1.1)$$

and we see from it that a relation exists between the structure of the medium and the electric field. This relation does not necessarily yield itself to the theoretical approach and is, generally found by experiment or by inference.

In this paper we consider an isotropic, homogeneous medium under isothermal conditions. In this case we have enough physical data on which to base an assumption that the equation of state can be represented as

$$\mathbf{j} = \sigma(j)\,\mathrm{E}, \qquad j = |\mathbf{j}| \qquad (1.2)$$

where **E** is the electric field strength vector and σ denotes the electric conductivity of the medium depending uniquely on the current density j.

To simplify the theoretical considerations that follow and to give a clearer picture of electrodynamic phenomena in a nonlinear medium, we shall only study twodimensional steady fields. Condition of the continuity of current and the law of induction, yield two equations

$$\frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} = 0, \quad \frac{\partial}{\partial y}\left(\frac{j_x}{\sigma}\right) - \frac{\partial}{\partial x}\left(\frac{j_y}{\sigma}\right) = 0 \qquad (1.3)$$

from which it follows that we can introduce the force stream function $Q(x, y)$ and the potential stream function $P(x, y)$ by means of

$$i_x = \frac{\partial Q}{\partial y} = \sigma \frac{\partial P}{\partial x}, \qquad i_y = -\frac{\partial Q}{\partial x} = \sigma \frac{\partial P}{\partial y} \tag{1.4}$$

When

$$\frac{\partial j}{\partial x} = \frac{1}{j}\left(i_x \frac{\partial i_x}{\partial x} + i_y \frac{\partial i_y}{\partial x}\right), \qquad \frac{\partial j}{\partial y} = \frac{1}{j}\left(i_x \frac{\partial i_x}{\partial y} + i_y \frac{\partial i_y}{\partial y}\right) \tag{1.5}$$

is taken into account, we obtain (1.3) as

$$\frac{\partial i_x}{\partial y} - \frac{\partial i_y}{\partial x} - \frac{1}{J^2}\left[i_x^2 \frac{\partial i_x}{\partial y} - i_x i_y \left(\frac{\partial i_x}{\partial x} - \frac{\partial i_y}{\partial y}\right) - i_y^2 \frac{\partial i_y}{\partial x}\right] = 0 \tag{1.6}$$

$J^2 = \sigma j (d\sigma/dj)^{-1}$ is the specific electric current density defining the degree of nonlinearity of the medium, or of the variation in conductivity. When the conductivity of the medium depends strongly on the current density j, then J assumes its smallest values, if on the other hand the nonlinearity disappears, then $J^2 \to \infty$.

Relations (1.6) and (1.3) yield the following expression for $Q(x, y)$

$$\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} - \frac{1}{J^2}\left[\left(\frac{\partial Q}{\partial x}\right)^2 \frac{\partial^2 Q}{\partial x^2} - 2\frac{\partial Q}{\partial x}\frac{\partial Q}{\partial y}\frac{\partial^2 Q}{\partial x \partial y} + \left(\frac{\partial Q}{\partial y}\right)^2 \frac{\partial^2 Q}{\partial y^2}\right] = 0 \tag{1.7}$$

To obtain an equation for $P(x, y)$ we use the following relations:

$$\left(1 - \frac{i_x^2}{J^2}\right)\frac{\partial i_x}{\partial x} - \frac{i_x i_y}{J^2}\frac{\partial i_y}{\partial x} = \sigma \frac{\partial^2 P}{\partial x^2}, \qquad \left(1 - \frac{i_y^2}{J^2}\right)\frac{\partial i_y}{\partial y} - \frac{i_x i_y}{J^2}\frac{\partial i_x}{\partial y} = \sigma \frac{\partial^2 P}{\partial y^2} \tag{1.8}$$

$$\left(1 - \frac{i_x^2}{J^2}\right)\frac{\partial i_x}{\partial y} - \frac{i_x i_y}{J^2}\frac{\partial i_y}{\partial y} = \sigma \frac{\partial^2 P}{\partial x \partial y}, \qquad \left(1 - \frac{i_y^2}{J^2}\right)\frac{\partial i_y^2}{\partial x} - \frac{i_x i_y}{J^2}\frac{\partial i_x}{\partial x} = \sigma \frac{\partial^2 P}{\partial x \partial y}$$

obtained from (1.4), together with the condition of continuity of the current (1.3). Simple transformations yield

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} - \frac{1}{J^2}\left[\left(\frac{\partial P}{\partial x}\right)^2 \frac{\partial^2 P}{\partial x^2} - 2\frac{\partial P}{\partial x}\frac{\partial P}{\partial y}\frac{\partial^2 P}{\partial x \partial y} + \left(\frac{\partial P}{\partial y}\right)^2 \frac{\partial^2 P}{\partial y^2}\right] = 0 \tag{1.9}$$

Quasilinear Eqs. (1.7) and (1.9) have the same external appearance and are of the mixed elliptic-hyperbolic type. The difference between them lies in the fact that expressions for the specific current density J in terms of P and Q are not the same, and that the differences between them are not necessarily simple.

If σ is independent of j (linear medium), then P and Q are harmonic functions

$$\Delta P(x, y) = 0, \qquad \Delta Q(x, y) = 0 \qquad (\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2) \tag{1.10}$$

This fact is widely used when methods of the theory of the complex variable are applied to the computation of electric fields.

Nonlinear Eqs. (1.7) and (1.9) are not easy to investigate and they are usually solved with help of various approximate methods. In particular, when the conductivity σ depends weakly on j and, as a result, J assumes large values, we can expand P and Q into series in terms of a small parameter $1/J^2$

$$P(x, y) = P_0(x, y) + \frac{1}{J^2}P_1(x, y) + \frac{1}{J^4}P_2(x, y) + \ldots \tag{1.11}$$

where the principal term of the series $P_0(x, y)$ corresponds to the solution of the linear conductivity, while the remaining terms $P_i(x, y)$ ($i = 1, 2, \ldots$) satisfy linear elliptic equations obtained by the substitution of (1.11) into (1.9).

2. Mapping onto the hodograph plane. We can transform exact nonlinear Eqs. (1.7) and (1.9) into exact linear equations by interchanging the dependent and independent variables. We either use the Legendre transformation, or introduce j and θ ($j = |\mathbf{j}|$, (θ is the angle between the vector $\mathbf{j}$ and the x-axis) as independent variables. Both methods are equivalent and their use in nonlinear electrodynamics is, in fact, analogous to the corresponding transformations in the theory of compressible fluid flows [1 and 2].

By (1.4) we can write

$$dQ = j_x\, dy - j_y\, dx, \qquad \sigma\, dP = j_x\, dx + j_y\, dy \tag{2.1}$$

from which it follows that

$$dz = (\sigma\, dP + i dQ)\,\frac{e^{i\theta}}{j}, \qquad j = (j_x^2 + j_y^2)^{1/2}, \quad \theta = \operatorname{arc\,tg}\frac{j_y}{j_x} \tag{2.2}$$

$$\frac{\partial z}{\partial j} = \left(\sigma\frac{\partial P}{\partial j} + i\frac{\partial Q}{\partial j}\right)\frac{e^{i\theta}}{j}, \qquad \frac{\partial z}{\partial \theta} = \left(\sigma\frac{\partial P}{\partial \theta} + i\frac{\partial Q}{\partial \theta}\right)\frac{e^{i\theta}}{j}$$

$$\left(D = \frac{\partial P}{\partial j}\frac{\partial Q}{\partial \theta} - \frac{\partial P}{\partial \theta}\frac{\partial Q}{\partial j} \equiv \frac{\partial (P, Q)}{\partial (j, \theta)} \neq 0\right)$$

Differentiating last two equations in θ and j respectively, gives

$$\frac{\partial Q}{\partial j} = \frac{\sigma}{j}\left(1 - \frac{j}{\sigma}\frac{d\sigma}{dj}\right)\frac{\partial P}{\partial \theta}, \qquad \frac{dQ}{\partial \theta} = -\sigma j\,\frac{\partial P}{\partial j} \tag{2.3}$$

which easily yield equations for P and Q in the hodograph plane

$$\frac{\partial^2 P}{\partial j^2} + \frac{1 - \Gamma^2}{j^2}\frac{\partial^2 P}{\partial \theta^2} + \frac{1 + \Gamma^2}{j}\frac{\partial P}{\partial j} = 0 \tag{2.4}$$

$$\left(\Gamma^2 = \frac{j}{\sigma}\frac{d\sigma}{dj}\right)$$

$$\frac{\partial^2 Q}{\partial j^2} + \frac{1 - \Gamma^2}{j^2}\frac{\partial^2 Q}{\partial \theta^2} + \frac{d}{dj}\left(\frac{j}{\sigma(1 - \Gamma^2)}\right)\frac{\partial Q}{\partial j} = 0 \tag{2.5}$$

Here $\Gamma^2 = j^2/J^2$ is a pure number characterizing the nonlinear conductivity. It plays an important role in the problems of electrodynamics of nonlinearly conducting media. In the following three cases of the distribution of current $\Gamma < 1$, $\Gamma = 1$ and $\Gamma > 1$. Eqs. (2.4) and (2.5) will be elliptic, parabolic and hyperbolic, respectively, We see that, depending on the type of our equation, the processes taking place in a nonlinear medium may differ appreciably. When $\Gamma < 1$, current distribution is defined by an elliptic equation and its solution can be obtained in terms of smooth functions. When $\Gamma > 1$, Eqs. (2.4) and (2.5) are hyperbolic and the solutions may exhibit discontinuities (on the characteristics). This points to the possibility of existence of electric shock waves in a nonlinear medium. Finally, if the conductivity σ is linearly dependent on the electric current density ($\sigma \equiv j$, $\Gamma = 1$), then (2.4), say, becomes parabolic

$$\frac{\partial^2 P}{\partial j^2} + \frac{2}{j}\frac{\partial P}{\partial j} = 0 \tag{2.6}$$

We note that Eqs. (2.4) to (2.6) are linear and that their solutions are, therefore much simpler than those of the corresponding nonlinear Eqs. (1.7) and (1.9) in the physical plane. We should however remember that when definite problems are being solved in the hodograph plane, boundary conditions become more complex and sometimes they cannot be obtained.

3. Nonlinear phenomena in plasma. Some energy relationships in the nonlinear plasma can be studied on the basis of the elementary theory. A number of physical problems arising in the studies of nonlinear conductivity in the plasma and computation of fields in such media (particularly in the cases when the conductivity is weakly dependent on the current density j), was investigated in [3 and 4].

When only elastic collisions are present, the energy balance for the electrons in an electric field per unit time, can be written as [5]

$$\mathbf{jE} = \frac{3}{2}\,\delta\,\frac{n_e}{\tau}\,k\,(T_e - T) \tag{3.1}$$

where n_e and τ are the concentration and effective time of collision of electrons; δ is the mean amount of energy transferred from an electron during its collision with heavy particles T and T_e denote the effective temperatures of heavy particles and electrons, respectively. Magnitudes δ, n_e and τ depend on T_e and are obtained either on the basis of the kinetic theory, or by suitable experiments. In particular, for weakly and fully ionised plasma we have $\delta = 2m_e/m_\alpha \sim 10^{-1}$ to 10^{-5} where m_α is the mass of a heavy particles and m_e is the mass of an electron.

Left-hand side of (3.1) denotes the work done by the field on the plasma in unit time, while the right-hand side denotes the energy lost by an electron in collisions with heavy particles.

Assuming that

$$\mathbf{j} = \sigma \mathbf{E}, \qquad \sigma = n_e e^2 \tau / m_e \tag{3.2}$$

we can write (3.1) as

$$j^2 = \frac{\delta}{2} n_e^2 e^2 \frac{3kT}{m_e} \left(\frac{T_e}{T} - 1 \right) \tag{3.3}$$

or, since $\mathbf{j} = n_e e \nu$ and $\delta = 2m_e / m_a$ (ν is the velocity vector of the oriented motion of electrons), as

$$v_{max}^2 = v_0^2 + v^2 \qquad \left(v_{max}^2 = \frac{3kT_e}{m_a}, \quad v_0^2 = \frac{3kT}{m_a} \right)$$
$$\frac{T_e}{T} = 1 + \Lambda^2 \qquad \left(\Lambda^2 = \frac{v^2}{v_0^2} \right) \tag{3.4}$$

Since $\nu_0{}^2$ is proportional to the local kinetic energy in the plasma and ν to the local electric field strength, the dimensionless parameter Λ characterizes the local ratio of the field strength to the thermal energy of the plasma. Moreover, ν_{max} denotes the maximum possible velocity of the oriented motion of electrons.

We shall now establish the character of nonlinearity in the plasma, by estimating the dimensionless parameters Γ and Λ. Returning to Expression (3.2) which gives the conductivity we find, that the electron concentration n_e and their relaxation time τ depend, in general, on T_e and therefore on the electric field. Three cases are possible: (a) n_e = const, $\tau(T_e)$; (b) $n_e(T_e)$, $\tau(T_e)$ and (c) $n_e(T_e)$, τ = const.

The first and simplest situation occurs, apparently, in the strongly ionised plasma. We can assume that

$$\tau = \tau_0 (T_e / T)^\gamma \tag{3.5}$$

where τ_0 is the time of relaxation when $T_e = T$ and γ is a parameter dependent on the mechanism of electron-electron collision, is approximately true. Then by the previous formulas we have the following expressions for the conductivity

$$\sigma = \sigma_0 (T_e / T)^\gamma = \sigma_0 (1 + \Lambda^2)^\gamma \tag{3.6}$$

for the specific current density

$$J^2 = \sigma j \left(\frac{d\sigma}{dj} \right)^{-1} = n_e^2 e^2 \frac{v_{max}^2}{2\gamma} = n_e^2 e^2 v_*^2 \tag{3.7}$$

and for the dimensionless number

$$\Gamma^2 = \frac{j^2}{J^2} = 2\gamma \frac{v^2}{v_{max}^2} = \frac{v^2}{v_*^2} \tag{3.8}$$

Here ν_* is the critical velocity of the electrons. If the oriented electron velocity is less than critical ($\nu < \nu_*$) then $\Gamma < 1$ and the solutions of (2.4) and (2.5) are smooth. On the other hand, when $\nu > \nu_*$, we have $\Gamma > 1$ and discontinuities in the electric field are quite possible. The latter can take place when the inequality $0 < \nu_* < \nu_{max} < \infty$ holds, which is possible whenever $\gamma > ½$ (e.g. when electrons collide with ions, we have $\gamma = 3/2$).

In two remaining cases the dependence of the electron concentration on T_e is quite complicated (for a weakly ionised plasma the Saha equation is applicable), therefore the interpretation of ν_* becomes less obvious.

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THE STABILITY OF THE STEADY - STATE SOLUTIONS IN THE THEORY OF THERMAL EXPLOSIONS

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S.A. KAGANOV
(Saratov)

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The steady-state theory of thermal explosion is concerned with solutions of the following boundary value problem [1 and 2]:

$$\Delta T + \varphi(T) = 0, \quad T|_\Gamma = 0 \tag{0.1}$$

where Γ is the surface enclosing the region G (a vessel), T is the temperature and $\varphi(T)$ is a positive, monotonously increasing function differentiable on $[0, +\infty]$. If a solution of (0.1) exists, then we assume that no explosion takes place in the vessel G, otherwise we assume that it does occur. In [1 and 2] the problem was studied for $\varphi(T) = e^T$ and the regions which possessed plane, cylindrical or spherical symmetry. In accordance with this the problem can be reduced to a problem for a segment, circle or a sphere with one independent variable equal to the distance from the center.

For a segment, the problem (0.1) becomes

$$\frac{d^2T}{dx^2} + \varphi(T) = 0, \qquad T|_{x=\pm h} = 0 \tag{0.2}$$

In [1 and 2] it was shown that a critical value $h = h_*$ exists for $\varphi(T) = e^T$ such, that when $0 \leqslant h \leqslant h_*$, a solution of (0.2) exists. When $h > h_*$, we have no solution, while when $0 < h < h_*$, we actually have two solutions. Denoting by $T_m = T(0)$ the maximum temperature and introducing the function $h = h(T_m)$, we obtain the corresponding curve as shown on Fig. 1. By symmetry we have $dT/dx = 0$ when $x = 0$ and the function $h(T_m)$ is singlevalued and continuous (a solution of the Cauchy's problem for (0.2) with conditions $dT/dx = 0$ and $T = T_m$ when $x = 0$ exists and depends continuously on T_m). In the case of a circle we have the analogous result. In the case of a sphere, a critical value of the radius exists also, but according to [2] the curve $h(T_m)$ is more complex. In [3] the problem was investigated for a function $\varphi(T)$ of the sufficiently general form and it was found that more than two solutions may exist for a given h, although uniqueness is not excluded. The curve $h(T_m)$ may have